

Invariances of the operator properties of frame multipliers under perturbations of frames and symbol

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Abstract

Let Φ and Ψ be frames for \mathcal{H} and let $M_{m,\Phi,\Psi}$ be a frame multiplier with the symbol m . In this paper, we restrict our investigation to show that the operator properties of $M_{m,\Phi,\Psi}$ are stable under the perturbations of Φ , Ψ and m . Also, special attention is devoted to the study of invertible frame multipliers. These results are not only of interest in their own right, but also they pave the way for obtaining some new results for Gabor multipliers which have been studied mostly by Hans Georg Feichtinger and his coauthors in recent years.

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1 Introduction

Throughout this paper, we denote by \mathcal{H} a separable Hilbert space with the inner product “ $\langle \cdot, \cdot \rangle$ ”. Also, ℓ^2 and ℓ^∞ have their usual meanings and $(\delta_n)_n$ refers to the canonical orthonormal basis of ℓ^2 . Moreover, our notation and terminology are standard and, concerning frames in Hilbert spaces, they are in general those of the book [4] of Christensen. All over in this paper Φ and Ψ are sequences $(\varphi_n)_n$ and $(\psi_n)_n$ in \mathcal{H} , and in the case where Φ is a Bessel sequence, the analysis operator is denoted by U_Φ , the synthesis operator by T_Φ and the frame operator by S_Φ . The canonical dual of the frame Φ is denoted by $\tilde{\Phi} = (\tilde{\varphi}_n)_n$, and we denote by A_Φ and B_Φ the lower and upper frame bounds of Φ , respectively. The notation m is used to denote a complex scalar sequence $(m_n)_n$, $1/m = (1/m_n)_n$ and $\overline{m} = (\overline{m}_n)_n$, where \overline{m}_n denotes the complex conjugate of m_n . The sequence m is called semi-normalized if $0 < \inf_n |m_n| \leq \sup_n |m_n| < \infty$. For $m \in \ell^\infty$, \mathcal{M}_m denotes the mapping defined by $\mathcal{M}_m(c_n)_n = (m_n c_n)_n$ from ℓ^2 into ℓ^2 . Also, the main object of study of this work is the operator $M_{m,\Phi,\Psi}$ which denotes the map defined by the equality

$$M_{m,\Phi,\Psi}(f) = T_\Phi \mathcal{M}_m U_\Psi(f) = \sum_{n=1}^{\infty} m_n \langle f, \psi_n \rangle \phi_n \quad (f \in \mathcal{H});$$

this operator is called multiplier with symbol m . In particular, the adjoint of $M_{m,\Phi,\Psi}$, denoted by $(M_{m,\Phi,\Psi})^*$, is equal to $M_{\overline{m},\Psi,\Phi}$.

The notions of Bessel multiplier, frame multiplier and Riesz multiplier, as an extension of Gabor multipliers [11], were introduced and first studied by Balazs [1] for Hilbert space. As far as we know the subject, the starting point of the study of such operators is Schatten's paper [14], and in the works [9, 10, 11], they are considered in Fourier and Gabor analysis. In particular, this class of operators has been extensively studied and has many applications in different contexts. The reader can find in the papers [1–3] and [5–12] a lot of information about the history of this class of operators, some of their properties and their applications in scientific disciplines and in modern life.

For some applications it is important to consider the stability of the operator properties of $M_{m,\Phi,\Psi}$ under perturbations of Φ , Ψ and m . For this purpose, we restrict our investigation to the study of the effect of perturbations of Φ , Ψ and m on the operator properties of $M_{m,\Phi,\Psi}$. Among other things, we obtain some conditions under which the inverse of an invertible frame multiplier can be represented as a multiplier with the reciprocal symbol and canonical dual frames of the given ones.

2 Main results

Let us commence with the following result, which provides some equivalent conditions for those invertible frame multipliers $M_{m,\Phi,\Psi}$ whose inverses is $M_{1/m,\tilde{\Psi},\tilde{\Phi}}$.

Theorem 2.1 *Suppose that Φ and Ψ are frames for \mathcal{H} , and that m is a semi-normalized sequence for which $M_{m,\Phi,\Psi}$ is invertible. Suppose also that A_Φ and A_Ψ are the optimal lower frame bounds of Φ and Ψ , respectively, and $|m|$ refers to the sequence $(|m_n|)_n$. Then $M_{m,\Phi,\Psi}^{-1} = M_{1/m,\tilde{\Psi},\tilde{\Phi}}$ if and only if one of the following conditions is satisfied:*

- (i) $\|S_\Psi^{-1}\| = \|M_{m,\Phi,\Psi}^{-1} T_\Phi \mathcal{M}_{|m|}\|^2$.
- (ii) *The optimal upper frame bound of the frame $(M_{m,\Phi,\Psi}^{-1}(m_n \varphi_n))_n$ is A_Ψ^{-1} .*
- (iii) $\|S_\Phi^{-1}\| = \|(M_{m,\Phi,\Psi}^{-1})^* T_\Psi \mathcal{M}_{|m|}\|^2$.
- (iv) *The optimal upper frame bound of the frame $(M_{\overline{m},\Psi,\Phi}^{-1}(\overline{m}_n \psi_n))_n$ is A_Φ^{-1} .*

Proof. Denote $M := M_{m,\Phi,\Psi}$ and suppose that

$$\psi_n^\dagger = M^{-1}(m_n \varphi_n) \quad \text{and} \quad \varphi_n^\dagger = (M^{-1})^*(\overline{m}_n \psi_n) \quad (n \in \mathbb{N}).$$

As was shown in [1, Theorem 1.1], $\Psi^\dagger = (\psi_n^\dagger)$ and $\Phi^\dagger = (\varphi_n^\dagger)$ are the unique dual frames of Ψ and Φ , respectively, such that

$$M^{-1} = M_{1/m,\Psi^\dagger,\tilde{\Phi}} \quad \text{and} \quad M^{-1} = M_{1/m,\tilde{\Psi},\Phi^\dagger}.$$

First observe that

$$S_{\Psi^\dagger} = M^{-1} T_\Phi \mathcal{M}_{|m|} (M^{-1} T_\Phi \mathcal{M}_{|m|})^* \quad \text{and} \quad S_{\tilde{\Psi}} = S_\Psi^{-1},$$

and

$$S_{\Phi^\dagger} = (M^{-1})^* T_\Psi \mathcal{M}_{|m|} \left((M^{-1})^* T_\Psi \mathcal{M}_{|m|} \right)^* \quad \text{and} \quad S_{\tilde{\Phi}} = S_{\Phi}^{-1}.$$

From these, by [13, Theorem 2.3.1], we have

$$\|S_{\Psi^\dagger}\| = \|M^{-1} T_\Phi \mathcal{M}_{|m|}\|^2 \quad \text{and} \quad \|S_{\Phi^\dagger}\| = \|(M^{-1})^* T_\Psi \mathcal{M}_{|m|}\|^2.$$

Hence, if $\Psi^\dagger = \tilde{\Psi}$ [respectively, $\Phi^\dagger = \tilde{\Phi}$], then the condition (i) [respectively, (iii)] is satisfied.

Conversely, if condition (i) is satisfied, then, by using [4, Theorem 5.7.4] and its proof, there exists a bounded operator $W : \ell^2 \rightarrow \mathcal{H}$ such that

$$\psi_n^\dagger = \tilde{\psi}_n + V \delta_n \quad (n \in \mathbb{N}), \quad (1)$$

where $V = W(Id_{\mathcal{H}} - U_\Psi S_\Psi^{-1} T_\Psi)$. Now, it is not hard to check that $T_{\Psi^\dagger} = T_{\tilde{\Psi}} + V$, $U_{\Psi^\dagger} = U_{\tilde{\Psi}} + V^*$, $T_{\tilde{\Psi}} V^* = 0$ and $V U_{\tilde{\Psi}} = 0$. It follows that $S_{\Psi^\dagger} = S_{\tilde{\Psi}} + V V^*$. This, together with the positivity of the operators S_{Ψ^\dagger} , $S_{\tilde{\Psi}}$ and $V V^*$ imply that

$$\|S_{\Psi^\dagger}\| = \sup_{\|f\| \leq 1} \langle S_{\Psi^\dagger} f, f \rangle = \|S_{\tilde{\Psi}}\| + \sup_{\|f\| \leq 1} \|V^* f\|^2. \quad (2)$$

Therefore, by Eq. (1) and (2), we get $V = 0$ and thus $\Psi^\dagger = \tilde{\Psi}$. Similarly, if condition (ii) is satisfied, the inverse of M is $M_{1/m, \tilde{\Psi}, \tilde{\Phi}}$ either.

Finally, to prove condition (ii) [respectively, (iv)] is equivalent to the equality $M^{-1} = M_{1/m, \tilde{\Psi}, \tilde{\Phi}}$, it will be enough to note that, by [4, Lemma 5.1.6 and Proposition 5.3.8], $\|S_{\tilde{\Psi}}\| = A_{\tilde{\Psi}}^{-1}$ [respectively, $\|S_{\tilde{\Phi}}\| = A_{\tilde{\Phi}}^{-1}$]; this is because of, as seen above $\|S_{\Psi^\dagger}\| = \|S_{\tilde{\Psi}}\|$ [respectively, $\|S_{\Phi^\dagger}\| = \|S_{\tilde{\Phi}}\|$] if and only if $\Psi^\dagger = \tilde{\Psi}$ [respectively, $\Phi^\dagger = \tilde{\Phi}$], and on the other hand $\|S_{\Psi^\dagger}\|$ [respectively, $\|S_{\Phi^\dagger}\|$] is equal to the optimal upper frame bound of Ψ^\dagger [respectively, Φ^\dagger]. ■

For the formulation of the following statements, which guarantee the stability of the operator properties of a frame multiplier under the perturbations of frames, we need the following definition. Some basic properties of multipliers can be found in [1, Theorem 6.1] and [11].

Definition 2.2 Let Φ be a sequence in \mathcal{H} , $m \in \ell^\infty$ and $\mu, \varepsilon > 0$.

- (i) We say that a sequence $\Phi' = (\varphi'_n)_n$ in \mathcal{H} is a μ -perturbation of Φ if $\|T_\Phi - T_{\Phi'}\| \leq \mu$.
- (ii) We call the sequence $m' = (m'_n)_n$ a ε -perturbation of m whenever $\|m - m'\|_\infty \leq \varepsilon$.

In what follows, for closed subspace X of ℓ^2 , the notation π_X is used to denote the orthogonal projection of ℓ^2 onto X . Moreover, the range of the operator Q is denoted by $\mathcal{R}(Q)$.

Theorem 2.3 Let Φ and Ψ be frames for \mathcal{H} with frame bounds A_Φ, B_Φ and A_Ψ, B_Ψ , respectively, and let m be a semi-normalized symbol. If Φ' is a μ -perturbation of Φ which $\mu < \sqrt{A_\Phi}$, then there exists a frame $\Psi' = (\psi'_n)_n$ which is a $\lambda\mu$ -perturbation of Ψ for some $\lambda > 0$ and $M_{m, \Phi', \Psi'} = M_{m, \Phi, \Psi}$. In particular, the operator properties of $M_{m, \Phi, \Psi}$ (such as compactness, invertibility, surjectivity and etc.) are stable under the perturbations of Φ .

Proof. First note that [4, Theorem 5.6.1] together with the fact that Φ' is a μ -perturbation of Φ , where $\mu < \sqrt{A_\Phi}$, implies that Φ' is a frame for \mathcal{H} with lower frame bound $A_{\Phi'} = (\sqrt{A_\Phi} - \mu)^2$. Hence, since m is a semi-normalized sequence, we can deduce that $m\Phi'$ is a frame for \mathcal{H} with lower frame bound $A_{m\Phi'} = (\inf_n |m_n|)^2 (\sqrt{A_\Phi} - \mu)^2$. In particular, it is easy to see that

$$\ell^2 = \mathcal{R}(U_{m\Phi'}) \oplus \ker(T_{m\Phi'}) \quad \text{and} \quad U_{m\Phi'} T_{m\Phi'}^\sim = U_{m\Phi'} S_{m\Phi'}^{-1} T_{m\Phi'} = \pi_{\mathcal{R}(T_{m\Phi'})}.$$

Now, if we set

$$\Psi' := \left(M_{\overline{m}, \Psi, \Phi} S_{m\Phi'}^{-1}(m_n \varphi'_n) + T_\Psi \pi_{\ker(T_{m\Phi'})}(\delta_n) \right)_n \quad (n \in \mathbb{N}),$$

then, for each sequence $c = (c_n) \in \ell^2$, we have

$$\begin{aligned} T_{\Psi'} c &= \sum_{n=1}^{\infty} c_n M_{\overline{m}, \Psi, \Phi} S_{m\Phi'}^{-1}(m_n \varphi'_n) + \sum_{n=1}^{\infty} c_n T_\Psi \pi_{\ker(T_{m\Phi'})}(\delta_n) \\ &= T_\Psi U_{m\Phi'} T_{m\Phi'}^\sim c + T_\Psi \pi_{\ker(T_{m\Phi'})} c. \end{aligned}$$

Therefore, we observe

$$\begin{aligned} T_\Psi (U_{m\Phi} - U_{m\Phi'}) T_{m\Phi'}^\sim &= T_\Psi U_{m\Phi} T_{m\Phi'}^\sim - T_\Psi U_{m\Phi'} T_{m\Phi'}^\sim + T_\Psi \pi_{\ker(T_{m\Phi'})} - T_\Psi \pi_{\ker(T_{m\Phi'})} \\ &= T_\Psi U_{m\Phi} T_{m\Phi'}^\sim + T_\Psi \pi_{\ker(T_{m\Phi'})} - (T_\Psi \pi_{\mathcal{R}(T_{m\Phi'})} + T_\Psi \pi_{\ker(T_{m\Phi'})}) \\ &= T_{\Psi'} - T_\Psi. \end{aligned}$$

It follows that

$$\begin{aligned} \|T_{\Psi'} - T_\Psi\| &\leq \|T_\Psi\| \|U_{m\Phi} - U_{m\Phi'}\| \|T_{m\Phi'}^\sim\| \\ &\leq \|T_\Psi\| \|m\|_\infty \|U_\Phi - U_{\Phi'}\| \|T_{m\Phi'}^\sim\| \\ &\leq \mu \frac{\|m\|_\infty \sqrt{B_\Psi}}{(\inf_n |m_n|)(\sqrt{A_\Phi} - \mu)}, \end{aligned}$$

and thus Ψ' is a $\lambda\mu$ -perturbation of Ψ , where $\lambda := \|m\|_\infty \sqrt{B_\Psi} / (\inf_n |m_n|)(\sqrt{A_\Phi} - \mu)$. Finally, we note that

$$\begin{aligned} M_{m, \Phi', \Psi'}(f) &= \sum_{n=1}^{\infty} \langle f, M_{\overline{m}, \Psi, \Phi} S_{m\Phi'}^{-1}(m_n \varphi'_n) \rangle m_n \varphi'_n + \sum_{n=1}^{\infty} \langle f, T_\Psi \pi_{\ker(T_{m\Phi'})}(\delta_n) \rangle m_n \varphi'_n \\ &= M_{m, \Phi, \Psi}(f), \end{aligned}$$

for all $f \in \mathcal{H}$. We have now completed the proof of the theorem. ■

The following remark is now immediate:

Remark 2.4 Let Φ and Ψ be frames for \mathcal{H} with frame bounds A_Φ, B_Φ and A_Ψ, B_Ψ , respectively, and let m be a semi-normalized symbol. With an argument similar to the proof of Theorem 2.3

and using the adjoint of the frame multiplier $M_{m,\Phi,\Psi}$ one can show that if Ψ' is a μ -perturbation of Ψ which $\mu < \sqrt{A_\Psi}$, then there exists a frame Φ' which is a $\lambda\mu$ -perturbation of Φ for some $\lambda > 0$ and $M_{m,\Phi',\Psi'} = M_{m,\Phi,\Psi}$. In particular, the operator properties of $M_{m,\Phi,\Psi}$ are stable under the perturbations of Ψ .

It is notable that a ε -perturbation of a sequence $m \in \ell^\infty$ is not necessarily a semi-normalized sequence even if m is semi-normalized. In the case where m is a sequence in ℓ^∞ for which the inequality $\inf_n |m_n| > 0$ is not necessarily valid, we have the following result. The reader will remark that a result similar to the following theorem, which is stated for μ -perturbation of Φ , can be formulated for μ -perturbation of Ψ , and so the details are omitted here.

Theorem 2.5 *Let Φ and Ψ be frames for \mathcal{H} with frame bounds A_Φ, B_Φ and A_Ψ, B_Ψ , respectively, and let m be a sequence in ℓ^∞ such that the frame multiplier $M_{m,\Phi,\Psi}$ is invertible. If Φ' is a μ -perturbation of Φ which $\mu\|m\|_\infty < (\sqrt{B_\Phi}\|M_{m,\Phi,\Psi}^{-1}\|)^{-1}$, then there exists a frame Ψ' which is a $\lambda\mu$ -perturbation of Ψ for some $\lambda > 0$ and $M_{m,\Phi',\Psi'} = M_{m,\Phi,\Psi}$. In particular, the operator properties of $M_{m,\Phi,\Psi}$ are stable under the perturbations of Φ .*

Proof. First note that, it is not hard to check that $m\Phi$ is a frame with lower frame bound $A_{m\Phi} := (B_\Phi\|M_{m,\Phi,\Psi}^{-1}\|^2)^{-1}$. Moreover, we observe that

$$\|T_{m\Phi} - T_{m\Phi'}\| \leq \|m\|_\infty \|T_\Phi - T_{\Phi'}\| \leq \mu\|m\|_\infty.$$

Hence, [4, Theorem 5.6.1] implies that $m\Phi'$ is a frame for \mathcal{H} , and thus if we set

$$\psi'_n := M_{\overline{m},\Psi,\Phi} S_{m\Phi'}^{-1}(m_n \varphi'_n) + T_\Psi \pi_{\ker(T_{m\Phi'})}(\delta_n) \quad (n \in \mathbb{N}),$$

then with an argument similar to the proof of Theorem 2.3 one can show that Ψ' is the desired frame. \blacksquare

Next we turn our attention to the perturbations of the symbol of a frame multiplier whose proof is omitted for conciseness, since it can be obtained with an argument similar to the proof of Theorems 2.3 and 2.5.

Theorem 2.6 *Let Φ and Ψ be frames for \mathcal{H} with frame bounds A_Φ, B_Φ and A_Ψ, B_Ψ , respectively, and let m' be a ε -perturbation of $m \in \ell^\infty$. If either*

- *the frame multiplier $M_{m,\Phi,\Psi}$ is invertible and $\varepsilon B_\Phi < 1/\|M_{m,\Phi,\Psi}^{-1}\|$;*
- *or m is semi-normalized and $\varepsilon\sqrt{B_\Phi} < (\inf_n |m_n|)\sqrt{B_\Phi}$.*

Then in both cases, $\Psi' := \left(M_{\overline{m},\Psi,\Phi} S_{m'\Phi}^{-1}(m'_n \varphi_n) + T_\Psi \pi_{\ker(T_{m'\Phi})}(\delta_n)\right)_n$ is a $\delta\varepsilon$ -perturbation of Ψ for some $\delta > 0$ and $M_{m',\Phi,\Psi'} = M_{m,\Phi,\Psi}$. In particular, the operator properties of $M_{m,\Phi,\Psi}$ are stable under the perturbations of m .

The next result gives a new representation for the inverse of any invertible frame multiplier with semi-normalized symbol. In particular, this proposition shows that the inverse of invertible frame multiplier $M_{m,\Phi,\Psi}$ has a decomposition into a sum of $M_{1/m,\tilde{\Psi},\tilde{\Phi}}$ and $\Gamma^*U_{\tilde{\Phi}}$, where Γ is uniquely determined.

Proposition 2.7 *Suppose that Φ and Ψ are frames for \mathcal{H} , and that the symbol m is semi-normalized. If $M_{m,\Phi,\Psi}$ is an invertible multiplier, then there exists a unique bounded operator $\Gamma : \mathcal{H} \rightarrow \ell^2$ such that*

$$M_{m,\Phi,\Psi}^{-1} = M_{1/m,\tilde{\Psi},\Phi^d} + \Gamma^*U_{\Phi^d},$$

for all dual frames $\Phi^d = (\varphi_n^d)_n$ of Φ .

Proof. Define $\Gamma : \mathcal{H} \rightarrow \ell^2$ by

$$\Gamma(f) := U_{\Phi}(M_{m,\Phi,\Psi}^{-1})^*(f) - \mathcal{M}_{1/\overline{m}}U_{\Psi}S_{\Psi}^{-1}(f) \quad (f \in \mathcal{H}). \quad (3)$$

Then it is not hard to check that the operator Γ is bounded. In particular, $M_{m,\Phi,\Psi}^{-1}T_{\Phi} = \Gamma^* + S_{\Psi}^{-1}T_{\Psi}\mathcal{M}_{1/m}$. Using any dual frame Φ^d of Φ we get

$$M_{m,\Phi,\Psi}^{-1} = S_{\Psi}^{-1}T_{\Psi}\mathcal{M}_{1/m}U_{\Phi^d} + \Gamma^*U_{\Phi^d}. \quad (4)$$

It follows that $M_{m,\Phi,\Psi}^{-1} = M_{1/m,\tilde{\Psi},\Phi^d} + \Gamma^*U_{\Phi^d}$ for all dual frames Φ^d of Φ . Finally, the proof will be completed by showing that the operator Γ is uniquely determined. To this end, suppose on the contrary that Eq. (4) are hold for two operators Γ_1 and Γ_2 . Hence, we have $\Gamma_1^*U_{\Phi^d} = \Gamma_2^*U_{\Phi^d}$ for all dual frames Φ^d of Φ . We now invoke part (i) of [2, Theorem 1.2] to conclude that $\Gamma_1 = \Gamma_2$. ■

The following remark is now immediate:

Remark 2.8 Suppose that Φ and Ψ are frames for \mathcal{H} , and that the symbol m is semi-normalized. If $M_{m,\Phi,\Psi}$ is an invertible multiplier, then

- (i) For operator Γ in Proposition 2.7 it is not hard to check that $T_{\Psi}\Gamma = 0$. It follows that in the case where Ψ is a Riesz basis, then $M_{m,\Phi,\Psi}^{-1} = M_{1/m,\tilde{\Psi},\Phi^d}$ for all dual frames Φ^d of Φ .
- (ii) It can be shown by routine calculations that if Ψ is equivalent to $m\Phi$, then for each dual frames Φ^d of Φ the inverse of $M_{m,\Phi,\Psi}$ is $M_{1/m,\tilde{\Psi},\Phi^d}$. Conversely, if $M_{m,\Phi,\Psi}^{-1} = M_{1/m,\tilde{\Psi},\Phi^d}$ for all dual frames Φ^d of Φ , then Proposition 2.7 implies that $\Gamma^*U_{\Phi^d} = 0$ for all dual frames Φ^d of Φ . This together with [2, Theorem 1.2(i)] implies that $\Gamma = 0$. From this, by Eq. (3), we deduce that $\mathcal{M}_{\overline{m}}U_{\Phi}(M_{m,\Phi,\Psi}^{-1})^* = U_{\Psi}S_{\Psi}^{-1}$. It follows that

$$\langle f, m_n\phi_n \rangle = \langle f, M_{m,\Phi,\Psi}S_{\Psi}^{-1}\psi_n \rangle \quad (f \in \mathcal{H} \text{ and } n \in \mathbb{N}),$$

and thus the frames Ψ and $m\Phi$ are equivalent.

(iii) With an argument similar to the proof of Proposition 2.7 above one can show that $\Theta = U_\Psi M_{m,\Phi,\Psi}^{-1} - \mathcal{M}_{1/m} U_\Phi S_\Phi^{-1}$ is the unique bounded operator such that

$$M_{m,\Phi,\Psi}^{-1} = M_{1/m,\Psi^d,\tilde{\Phi}} + T_{\Psi^d} \Theta,$$

for all dual frames Ψ^d of Ψ . In particular, $T_\Phi \Theta = 0$. Hence, if Φ is a Riesz basis, then $M_{m,\Phi,\Psi}^{-1} = M_{1/m,\Psi^d,\tilde{\Phi}}$ for all dual frames Ψ^d of Ψ . Moreover, for each dual frames Ψ^d of Ψ $M_{m,\Phi,\Psi}^{-1} = M_{1/m,\Psi^d,\tilde{\Phi}}$ if and only if Φ is equivalent to $\overline{m}\Psi$.

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